8.2

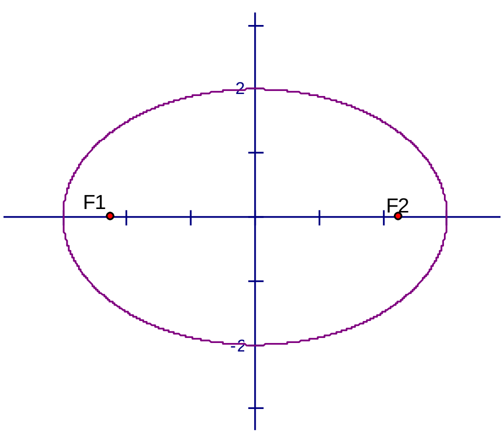
Ellipses

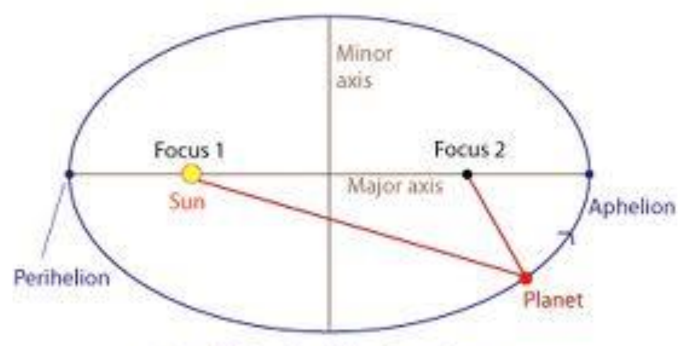
An **ellipse** is the collection of all points in the plane the sum of whose distances from **two fixed points** (**foci**) is a constant.

The line containing the foci is the **major axis.**

The midpoint of the line segment joining the foci is the **center of the ellipse.**

The line through the center and perpendicular to the major axis is the **minor axis.**





An elliptical orbit of a planet
(greatly exaggerated)

The two points of intersection of the ellipse and the major axis are the **vertices, V_1 and V_2** of the ellipse. The distance from one vertex to the other is the **length of the major axis.** The ellipse is symmetric with respect to its major axis, with respect to its minor axis, and with respect to its center.

If c is the distance from the center to a focus, then one focus will be at F_1= (-*c*,0) and the other at F_2 = (*c*, 0)

Equation of an Ellipse: Center at (0,0); Foci at ($\pm c$,0); Major Axis along the *x*-axis.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad where\ a>b>0\ and\ a^2=b^2+c^2$$

The major axis is the x-axis. The vertices are at (-*a*,0) and (*a*, 0) .

Find the equation of an ellipse with center (0,0), one focus at (3,0) and the vertex at (-4,0).

Equation of an Ellipse: Center at (0,0); Foci at (0, $\pm c$); Major Axis along the *y*-axis.

$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1 \qquad where\ a>b>0\ and\ a^2=b^2+c^2$

The major axis is the y-axis; the vertices are at (0,*a*) and (0,-*a*).

Ellipses with **Center at (*h, k*)** and Major Axis Parallel to a Coordinate Axis

Center	Major Axis	Foci	Vertices	equation
(h, k)	Parallel to x-axis	$(h + c, k)$	$(h + a, k)$	$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$
(h, k)	Parallel to x-axis	$(h - c, k)$	$(h - a, k)$	$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$
(h, k)	Parallel to y-axis	$(h, k + c)$	$(h, k + a)$	$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$
(h, k)	Parallel to y-axis	$(h, k - c)$	$(h, k - a)$	$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$

The eccentricity of an ellipse is $e=\frac{c}{a}=\frac{\sqrt{a^2-b^2}}{a}$ where *a* is the semimajor axis and *b* is the semiminor axis.

Examples:

Find an equation in standard form for the ellipse that satisfies the given conditions.

1) Foci $\left(\pm 2, 0\right)$, major axis length 10.

c = 2 and a = 10/2 = 5, so b = $\sqrt{a^2 - c^2} = \sqrt{25-4} = \sqrt{21}$

therefore the equation is $\frac{x^2}{25} + \frac{y^2}{21} = 1$.

2) Major axis endpoints $\left(\pm 5, 0\right)$, minor axis length 4.

b = 4/2 = 2, a = 5 so, the equation is $\frac{x^2}{25} + \frac{y^2}{4} = 1$.

3)The foci are (1, -4) and (5, -4); the major axis endpoints are (0, -4) and (6, -4).

We need to find (h, k) and a and b to write the equation.

The center (h, k) is the midpoint of (1, -4) and (5, -4) which is **(3, -4)**.

a is half the length of the major axis which is half the distance from (0, -4) and (6, -4), which is **3**.

c is half the distance between the foci (1, -4) and (5, -4), which is **2**.

$b = \sqrt{a^2 - c^2} = \sqrt{5}$ therefore, the equation is $\frac{\left(x-3\right)^2}{9} + \frac{\left(y+4\right)^2}{5} = 1$.

Prove that the graph of the equation is an ellipse. Find the vertices, foci, and eccentricity.

$3x^2+5y^2-12x+30y+42=0$ (Rearrange the equation putting x and y terms together and moving the constant to the right side of the equation.)

$3x^2-12x+5y^2+30y=-42$ (Complete the square for the x and y terms.)

$3(x^2-4x+4)+5(y^2+6y+9)=-42+12+45$ (Write equation in perfect square form.)

$3(x-2)^2+5(y+3)^2=15$ (Divide by 15 to get right side equal to 1.)

$$\frac{(x-2)^2}{5}+\frac{(y+3)^2}{3}=1$$

Vertices: $\left(2\pm\sqrt{5},-3\right)$, Foci: $\left(2\pm\sqrt{2},-3\right)$, Eccentricity: $\frac{\sqrt{2}}{\sqrt{5}}\approx 0.6325$